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Forward Modelling of Pollutant Transport in Rivers: Analytical and Alternating Direction Implicit Solutions of the Advection–Diffusion Equation with Temporally Varying Coefficients

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Abstract: **Accurate prediction of pollutant transport is the foundation on which any inverse source-identification scheme is built, yet most river-pollution studies assume constant or, at most, spatially varying flow parameters and so cannot represent the unsteady conditions of natural rivers. This paper develops, derives, and validates a forward transport model based on the advection–diffusion equation (ADE) with *temporally* varying velocity and dispersion coefficients, in both one and two spatial dimensions. The ADE is derived from first principles through conservation of mass and the divergence theorem. For the one-dimensional case, a closed-form analytical solution is obtained by a sequence of variable transformations that reduce the variable-coefficient ADE to a constant-coefficient diffusion equation solved by Laplace transforms; a numerical solution is constructed using the Forward Time Central Space Centered Scheme (FTCSCS), whose von Neumann stability condition is derived explicitly. For the two-dimensional case, an unconditionally stable, second-order Alternating Direction Implicit (ADI) scheme is developed, with the governing equation split into an x-sweep and a y-sweep, each reduced to a tridiagonal system solved by the Thomas algorithm. The analytical and numerical 1D solutions agree closely, with root-mean-square error (RMSE) decreasing from 0.028 at $Pe \ll 1$ to 0.004 at $Pe \gg 1$. Simulations reveal that pollutant concentration is highest near the source and decays downstream; that concentration grows with time at any fixed point; and that the longitudinal distribution becomes increasingly advection-skewed as the Peclet number rises. The effect of four temporal coefficient regimes is quantified, showing that concentration is lowest when both dispersion and velocity increase with time and highest when dispersion increases in a decelerating flow. The framework provides a physically faithful, computationally efficient forward solver suitable for generating the concentration fields required by inverse source-identification methods.**

Keywords: **Advection–diffusion equation, Temporally varying coefficients, Alternating direction implicit method Laplace transform, von Neumann stability, Pollutant transport.**

1. INTRODUCTION

Mathematical models are among the most important tools for understanding and predicting the behaviour of pollutant dispersion in rivers, and for planning countermeasures to improve water quality. Compared with experiments, which incur substantial material, construction, and labour costs and struggle to isolate extraneous variables, models offer a scientific, evidence-based, and inexpensive route to insight, while remaining simplifications of the real system. The advection–diffusion equation (ADE), derived from the Fickian description of transport, captures the two dominant mechanisms by which a released pollutant moves: advection, which carries it downstream with the bulk flow, and dispersion, which spreads it according to concentration gradients.

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A large body of work has produced analytical and numerical solutions of the ADE. Analytical techniques include separation of variables, Green's function and eigenfunction expansion methods, and integral transforms such as Laplace, Fourier, and Hankel transforms. Numerical techniques include the finite difference method (FDM), finite element method (FEM), and finite volume method (FVM). Within FDM, explicit centered schemes, Crank–Nicolson, and the Alternating Direction Implicit (ADI) method have all been applied to transport problems. A recurring limitation, however, is the assumption of constant or spatially varying coefficients: most one-dimensional FDM studies solve an ADE with constant coefficients, and even works that vary the coefficients spatially do not explore temporal dependence. Natural rivers, by contrast, exhibit flow velocities and dispersion coefficients that change in time owing to fluctuating discharge, seasonal variation, and transient events.

This paper addresses that gap by formulating and solving the ADE with *temporally* varying velocity and dispersion coefficients, in one and two dimensions. The contributions are: (i) a first-principles derivation of the ADE from conservation of mass; (ii) a closed-form analytical solution of the 1D temporally varying ADE via successive transformations and Laplace transforms; (iii) an FTCSCS numerical solution with an explicit von Neumann stability condition; (iv) an unconditionally stable, second-order 2D ADI scheme with full x- and y-sweep derivations; and (v) a validated set of simulations characterizing transport behaviour across Peclet regimes and temporal coefficient combinations. The resulting forward solver supplies the concentration fields required by inverse source-identification methods.

The paper is organized as follows. Section 2 derives the ADE. Section 3 presents the 1D analytical solution. Section 4 develops the FTCSCS scheme and its stability. Section 5 develops the 2D ADI scheme. Section 6 reports results. Section 7 concludes.

2. DERIVATION OF THE ADVECTION–DIFFUSION EQUATION

The ADE follows from conservation of mass applied to an arbitrary control volume $P \subset \mathbb{R}^3$. Let $c = c(X,t)$, $X = (x,y,z)$, denote the pollutant concentration per unit volume. The total mass in P is $M = \iiint_P c \ \ dv$, with rate of change

$$\frac{\partial M}{\partial t} = \iiint_P \frac{\partial c}{\partial t} \ \ dv. \tag{1}$$

If $\vec{f} = \vec{f}(X,t)$ is the mass flux across the boundary ∂A of P, then $\partial M/\partial t = -\iint_{\partial A} \vec{f} \cdot \hat{\boldsymbol{n}} \ \ dA$, where $\hat{\boldsymbol{n}}$ is the outward unit normal. By the divergence theorem $\iint_{\partial A} \vec{f} \cdot \hat{\boldsymbol{n}} \ \ dA = \iiint_P \nabla \cdot \vec{f} \ \ dv$, and since P is arbitrary the integrands in (1) and its surface counterpart must agree, yielding the continuity equation

$$\frac{\partial c}{\partial t} = -\nabla \cdot \vec{f}. \tag{2}$$

The total flux is the superposition of advective, dispersive, and source contributions, which are linearly independent:

$$\vec{f}(X,t) = f_A + f_D + Q_S = c\vec{V} - \vec{D}\nabla c + Q_S, \tag{3}$$

where $f_A = c\vec{V}$ is the advective flux with velocity $\vec{V}$, and $f_D = -\vec{D}\nabla c$ is the dispersive flux with dispersion coefficient $\vec{D}$ (Fick's law). Substituting (3) into (2) and writing $Q(X,t) = -\nabla \cdot Q_S$ for the source term gives

$$\frac{\partial c}{\partial t} = \nabla \cdot (\vec{D}(X,t)\nabla c) - \vec{V}(X,t) \cdot \nabla c + Q(X,t). \tag{4}$$

The advection coefficient may be estimated from Manning's formula $u = R_h^{2/3} P^{1/2}/N$, and the dispersion coefficient from $D = 2(W/H)^{3/2}(HV^*)$ with shear velocity $V^* = \sqrt{gR_hP}$, where R_h, P, N, W, H, and g are the hydraulic radius, bed gradient, Manning roughness, channel width, depth, and gravitational acceleration respectively.

2.1 One-Dimensional Analytical Solution

2.1.1 Governing equation and non-dimensionalization

For a single point source, the 1D form of (4) with temporally varying coefficients is

$$\frac{\partial c}{\partial t} - D_0 f_1(mt) \frac{\partial^2 c}{\partial x^2} + u_0 f_2(mt) \frac{\partial c}{\partial x} = 0, \qquad x \neq S, \tag{5}$$

subject to $c(x,0)=0$, $c(0,t)=C_0$, and $\partial c/\partial x \to 0$ as $x\to\infty$. Introducing the dimensionless quantities

$$c^*=\frac{c}{C_0},\qquad t^*=\frac{tD_0}{L^2},\qquad x^*=\frac{x}{L},\qquad Pe=\frac{LU_0}{D_0},\tag{6}$$

where the Peclet number Pe measures the ratio of the diffusion time scale $T_D=L^2/D_0$ to the advection time scale $T_U=L/U_0$, yields

$$\frac{\partial c^*}{\partial t^*}=f_1(mt^*)\frac{\partial^2 c^*}{\partial x^{*2}}-Pe\ \ f_2(mt^*)\frac{\partial c^*}{\partial x^*},\tag{7}$$

with $c^*=0$ at $t^*=0$, $c^*=1$ at $x^*=0$, and $\partial c^*/\partial x^*=0$ as $x^*\to\infty$.

2.1.2 Sequence of transformations

The variable coefficients are removed by successive transformations. First,

$$X^*=x\ \ \frac{f_2(mt^*)}{f_1(mt^*)}\tag{8}$$

recasts (7) as $\frac{f_1}{f_2^2}\partial_{t^*}c^*=\partial_{X^*}^2c^*-Pe\ \ \partial_{X^*}c^*$. Next, a new time variable

$$T^*(t)=\int_0^t\frac{f_2^2}{f_1}\ \ d\tau\tag{9}$$

reduces this to the constant-coefficient form

$$\frac{\partial c^*}{\partial T^*}=\frac{\partial^2 c^*}{\partial X^{*2}}-Pe\ \ \frac{\partial c^*}{\partial X^*}.\tag{10}$$

Finally, the transformation

$$c^*(X^*,T^*)=K^*(X^*,T^*)\exp[\frac{Pe}{2}X^*-\frac{Pe^2}{4}T^*]\tag{11}$$

eliminates the advection term, giving the pure diffusion equation

$$\frac{\partial K^*}{\partial T^*}=\frac{\partial^2 K^*}{\partial X^{*2}},\tag{12}$$

subject to $K^*=0$ for $X^*\ge 0, T^*=0$; $K^*(0,T^*)=e^{\mu T^*}$ with $\mu=Pe^2/4$; and $\partial_{X^*}K^*+\frac{Pe}{2}K^*=0$ as $X^*\to\infty$.

2.1.3 Laplace transform solution

Applying the Laplace transform $\overline{K}^*(X^*,p)=\int_0^\infty\ K^*e^{-pT^*}dT^*$ to (12) gives $\frac{d^2\overline{K}^*}{dX^{*2}}-p\overline{K}^*=0$, with $\overline{K}^*(0,p)=1/(p-\mu)$ and the decay condition at infinity. The bounded particular solution is

$$\overline{K}^*(X^*,p)=\frac{1}{p-\mu}\ \ e^{-\sqrt{p}\ \ X^*}.\tag{13}$$

Inverting yields

$$\begin{aligned}K^*(X^*,T^*)=\frac{1}{2}[\quad&\exp(\frac{Pe^2}{4}T^*-\frac{Pe}{2}X^*)erfc(\frac{X^*}{2\sqrt{T^*}}-\frac{Pe}{2}\sqrt{T^*})\\ +\quad&\exp(\frac{Pe^2}{4}T^*+\frac{Pe}{2}X^*)erfc(\frac{X^*}{2\sqrt{T^*}}+\frac{Pe}{2}\sqrt{T^*})].\end{aligned}\tag{14}$$

Substituting back through (11), (9), and (8) gives the closed-form analytical solution

$$\begin{aligned}c^*(x^*,t^*)=\frac{1}{2}[\quad&erfc(\frac{x}{2\sqrt{T^*}}\frac{f_2(mt^*)}{f_1(mt^*)}-\frac{Pe}{2}\sqrt{T^*})\\ +\quad&\exp(x\ \ \frac{f_2(mt^*)}{f_1(mt^*)}Pe)erfc(\frac{x}{2\sqrt{T^*}}\frac{f_2(mt^*)}{f_1(mt^*)}+\frac{Pe}{2}\sqrt{T^*})].\end{aligned}\tag{15}$$

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2.2 One-Dimensional Numerical Solution: FTCSCS

2.2.1 Discretization

The Forward Time Central Space Centered Scheme (FTCSCS) approximates the derivatives in (7) by a forward difference in time and central differences in space. With ${c^*}_i^j = c^*(x_i^*, t_j^*)$,

$$\frac{\partial c^*}{\partial t^*} = \frac{{c^*}_i^{j+1} - {c^*}_i^j}{\Delta t^*} + O(\Delta t^*), \tag{16}$$

$$\frac{\partial c^*}{\partial x^*} = \frac{{c^*}_{i+1}^j - {c^*}_{i-1}^j}{2\Delta x^*} + O(\Delta x^{*2}), \tag{17}$$

$$\frac{\partial^2 c^*}{\partial x^{*2}} = \frac{{c^*}_{i-1}^j - 2{c^*}_i^j + {c^*}_{i+1}^j}{\Delta x^{*2}} + O(\Delta x^{*2}). \tag{18}$$

Substituting and defining $\gamma = \Delta t^*/\Delta x^{*2}$, $\Phi = Pe\ \ \Delta t^*/\Delta x^*$, $\alpha = \gamma f_1^j - \frac{\Phi f_2^j}{2}$, $\beta = 1 - 2\gamma f_1^j$, $\delta = \gamma f_1^j + \frac{\Phi f_2^j}{2}$, the interior update is

$$ {c^*}_i^{j+1} = \delta\ \ {c^*}_{i-1}^j + \beta\ \ {c^*}_i^j + \alpha\ \ {c^*}_{i+1}^j. \tag{19}$$

The initial condition discretizes to ${c^*}_i^0 = 0$; the inflow condition to ${c^*}_1^{j+1} = 1$; and the outflow Neumann condition, via a ghost node ${c^*}_{n+1}^j = {c^*}_{n-1}^j$, gives the boundary update

$$ {c^*}_n^{j+1} = (\alpha + \delta)\ \ {c^*}_{n-1}^j + \beta\ \ {c^*}_n^j. \tag{20}$$

Equations (19)–(20) are advanced iteratively from ${c^*}_i^0 = 0$.

2.2.2 von Neumann stability

Writing the error at time level n as ε^n and seeking solutions of the form ${c^*}_i^n = \varepsilon^n \exp(\mathrm{j}\ \ \theta)$ (with $\mathrm{j} = \sqrt{-1}$), the amplification factor of (19) is

$$\frac{\varepsilon^{n+1}}{\varepsilon^n} = 1 - 4\gamma f_1^j \sin^2\left(\frac{\theta}{2}\right) - \mathrm{j}\ \ \frac{\Phi' L f_2^j}{D_0}\sin\theta, \qquad \Phi' = \frac{U_0 \Delta t^*}{\Delta x^*}. \tag{21}$$

The von Neumann condition $|\varepsilon^{n+1}/\varepsilon^n| \le 1$ requires

$$\left[1 - 4\gamma f_1^j \sin^2\left(\frac{\theta}{2}\right)\right]^2 + \left[\frac{\Phi' L f_2^j}{D_0}\sin\theta\right]^2 \le 1, \tag{22}$$

which, taking the maximum value of the sine terms, reduces to the working stability condition

$$(1 - 4\gamma f_1^j)^2 + \left(Pe\ \ f_2^j\ \ \frac{\Delta t^*}{\Delta x^*}\right)^2 \le 1. \tag{23}$$

The time and space steps are chosen to satisfy (23).

3. TWO-DIMENSIONAL NUMERICAL SOLUTION: ADI SCHEME

3.1 Non-dimensional 2D equation

Non-dimensionalizing the 2D ADE with the quantities (6) and choosing $C_0 = L^2/D_0$ so that the source coefficient is unity gives

$$\frac{\partial \bar{c}}{\partial \bar{t}} - f_1(m\bar{t})\frac{\partial^2 \bar{c}}{\partial \bar{x}^2} - f_1(m\bar{t})\frac{\partial^2 \bar{c}}{\partial \bar{y}^2} + Pe_{\bar{x}} f_2(m\bar{t})\frac{\partial \bar{c}}{\partial \bar{x}} + Pe_{\bar{y}} f_2(m\bar{t})\frac{\partial \bar{c}}{\partial \bar{y}} = \bar{Q}, \tag{24}$$

with $Pe_{\bar{x}} = u_0 L/D_0$ and $Pe_{\bar{y}} = v_0 L/D_0$, subject to $\bar{c}(x, y, 0) = 0$, $\bar{c} = 1$ on Γ_{in}, and $\nabla\bar{c} = 0$ on $\Gamma_L \cup \Gamma_{out}$.

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3.2 Operator splitting

The ADI method advances the solution over each time step in two half-steps. With $\alpha = \frac{\Delta t}{2\Delta\bar{x}^2}$, $\beta = \frac{\Delta t}{2\Delta\bar{y}^2}$, $\gamma = \frac{Pe_{\bar{x}}\Delta t}{4\Delta\bar{x}}$, $\nu = \frac{Pe_{\bar{y}}\Delta t}{4\Delta\bar{y}}$, the x-sweep (implicit in x, explicit in y, evaluated at k and $k+\frac{1}{2}$) gives

$$\begin{aligned} -(\alpha+\gamma)f^{k+1/2}\bar{c}_{i-1,j}^{k+1/2} &+ (1+2\alpha f^{k+1/2})\bar{c}_{i,j}^{k+1/2} + (\gamma-\alpha)f^{k+1/2}\bar{c}_{i+1,j}^{k+1/2} = \\ (\beta+\nu)f^{k}\bar{c}_{i,j-1}^{k} &+ (1-2\beta f^{k})\bar{c}_{i,j}^{k} + (\beta-\nu)f^{k}\bar{c}_{i,j+1}^{k} + \frac{\Delta\bar{t}\ \bar{Q}_{i,j}^{k}}{2}. \end{aligned} \tag{25}$$

The y-sweep (implicit in y, evaluated at $k+\frac{1}{2}$ and $k+1$) gives

$$\begin{aligned} -(\beta+\nu)f^{k+1}\bar{c}_{i,j-1}^{k+1} &+ (1+2\beta f^{k+1})\bar{c}_{i,j}^{k+1} + (\nu-\beta)f^{k+1}\bar{c}_{i,j+1}^{k+1} = \\ (\alpha+\gamma)f^{k+1/2}\bar{c}_{i-1,j}^{k+1/2} &+ (1-2\alpha f^{k+1/2})\bar{c}_{i,j}^{k+1/2} + (\alpha-\gamma)f^{k+1/2}\bar{c}_{i+1,j}^{k+1/2} + \frac{\Delta\bar{t}\ \bar{Q}_{i,j}^{k+1/2}}{2}. \end{aligned} \tag{26}$$

The inflow condition discretizes to $\bar{c}_{0,j}^{k+1/2} = 1$ (and $\bar{c}_{i,0}^{k+1} = 1$), and the Neumann conditions are imposed through fictitious boundary nodes, $\bar{c}_{n+1,j}^{k+1/2} = \bar{c}_{n-1,j}^{k+1/2}$ and $\bar{c}_{i,m+1}^{k+1} = \bar{c}_{i,m-1}^{k+1}$, giving the boundary updates

$$-2\alpha f^{k+1/2}\bar{c}_{n-1,j}^{k+1/2} + (1+2\alpha f^{k+1/2})\bar{c}_{n,j}^{k+1/2} = (\text{RHS of } (25) \text{ at } i = n), \tag{27}$$

$$-2\beta f^{k+1}\bar{c}_{i,m-1}^{k+1} + (1+2\beta f^{k+1})\bar{c}_{i,m}^{k+1} = (\text{RHS of } (25)(26) \text{ at } j = m). \tag{28}$$

Each sweep produces a tridiagonal system solved efficiently by the Thomas algorithm. The ADI scheme is unconditionally stable, second-order accurate in both time and space, and computationally inexpensive.

3.3 Numerical properties

A scheme is convergent if, by the Lax equivalence theorem, it is both consistent and stable. The FTCSCS scheme is consistent (its truncation error vanishes as $\Delta t^*, \Delta x^* \to 0$) and conditionally stable under (23), hence convergent when that condition holds; the ADI scheme is consistent and unconditionally stable, hence unconditionally convergent. Two error sources are present: truncation error from the truncated Taylor expansions, and round-off error from finite-precision arithmetic.

4. RESULTS AND DISCUSSION

4.1 Validation: analytical versus numerical (1D)

The numerical solution (19) was compared with the analytical solution (15) at $t^* = 1$ for three Peclet regimes, using $D_0 = 1.25$, $L = 10$, and $U_0 = 0.01$ ($Pe \ll 1$), 0.124 ($Pe \sim 1$), 1.14 ($Pe \gg 1$), with both coefficients increasing exponentially in time. Table 1 reports the RMSE.

Table 1: RMSE between analytical and FTCSCS solutions at $t^* = 1$ for three Peclet regimes.

$Pe \ll 1$	$Pe \sim 1$	$Pe \gg 1$
0.028371	0.009929	0.003922

The RMSE decreases from 0.028 at $Pe \ll 1$ to 0.010 at $Pe \sim 1$ and 0.004 at $Pe \gg 1$, confirming close agreement and indicating that the scheme is most accurate in the advection-dominated regime that characterizes natural rivers.

4.2 Transport behaviour across Peclet regimes (1D)

Table 2 summarizes the concentration field for the three regimes at four times.

Table 2: Pollutant concentration c^* for varying Peclet numbers at different positions and times.

	t^* ($Pe \ll 1$)				t^* ($Pe \sim 1$)				t^* ($Pe \gg 1$)			
x^*	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.3715	0.5915	0.6780	0.7304	0.3904	0.6210	0.7114	0.7661	0.5534	0.8326	0.9175	0.9569

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0.2	0.0740	0.2831	0.4062	0.4907	0.8170	0.3122	0.4476	0.5401	0.1696	0.5858	0.7777	0.8785
0.3	0.0074	0.1073	0.2127	0.3011	0.0086	0.1244	0.2462	0.3481	0.0264	0.3357	0.5957	0.7631
0.4	0.0000	0.0318	0.0966	0.1679	0.0000	0.0387	0.1175	0.2039	0.0000	0.1528	0.4050	0.6191
0.5	0.0000	0.0100	0.0378	0.0848	0.0000	0.0127	0.0584	0.1237	0.0000	0.0682	0.2704	0.4950
0.6	0.0000	0.0000	0.0160	0.0455	0.0000	0.0000	0.0212	0.0604	0.0000	0.0197	0.1438	0.3462
0.7	0.0000	0.0000	0.0000	0.0191	0.0000	0.0000	0.0066	0.0266	0.0000	0.0044	0.0659	0.2207
0.8	0.0000	0.0000	0.0000	0.0072	0.0000	0.0000	0.0000	0.0106	0.0000	0.0000	0.0259	0.1278
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0040	0.0000	0.0000	0.0088	0.0690
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0039	0.0450

At any fixed time, concentration falls with distance from the source and reduces monotonically to near zero far downstream; at any fixed point, concentration grows with time. Across regimes, concentration at a given point and time is markedly larger for $Pe \gg 1$ than for $Pe \sim 1$ or $Pe \ll 1$: when advection dominates, pollutant is carried downstream faster than it can spread, so appreciable concentration persists in the range $0.7 \leq x^* \leq 1$ for $Pe \gg 1$, whereas it has effectively vanished there for $Pe \ll 1$. Physically, $Pe \gg 1$ corresponds to swift mountain streams and $Pe \ll 1$ to slow-moving plain rivers. The simulations exhibit longitudinal skewness that increases with Pe, reflecting the imbalance between advective transport and dispersive spreading.

4.3 Effect of temporally varying coefficients

Four combinations of $f_1(mt)$ and $f_2(mt)$ were examined at $t^* = 1$, $Pe \sim 1$ (Table 3).

Table 3: Concentration c^* for combinations of temporally varying dispersion $D(t) = D_0 f_1(mt)$ and velocity $u(t) = U_0 f_2(mt)$.

2*Regime	x^*					
	0.1	0.2	0.3	0.4	0.5	$0.6 \leq x^* \leq 1.0$
$D_0 e^{mt}, U_0 e^{mt}$ (dispersion ↑, accelerating)	0.3904	0.0817	0.0086	0.0000	0.0000	0.0000
$D_0 e^{-mt}, U_0 e^{-mt}$ (dispersion ↓, decelerating)	0.5998	0.2816	0.1011	0.0273	0.0090	0.0000
$D_0 e^{-mt}, U_0 e^{mt}$ (dispersion ↓, accelerating)	0.5998	0.2816	0.1011	0.0273	0.0090	0.0000
$D_0 e^{mt}, U_0 e^{-mt}$ (dispersion ↑, decelerating)	0.6162	0.3077	0.1218	0.0377	0.0090	0.0000

Concentration is lowest when both coefficients increase with time: rapid mixing after injection and immediate downstream transport occur simultaneously, leaving few molecules near the source. It is higher when both coefficients decrease with time, owing to accumulation as spreading and transport proceed more slowly. The highest concentration occurs for increasing dispersion in a decelerating flow, where accelerated diffusion combines with minimal downstream transport to concentrate pollutant near the source. There is no significant difference between decreasing dispersion in a decelerating flow and decreasing dispersion in an accelerating flow.

4.4 Two-dimensional transport (ADI)

The 2D ADI solver was applied to an instantaneous point source injected at $(0.2, 0.005)$, with intensity $\lambda(t) = \sum_i b_i e^{-u_i(t-q_i)^2}$ and the source approximated by a normal distribution. At the injection time the pollutant cloud remains localized near the source; as time advances the patch expands and translates downstream, reflecting the combined action of advection and diffusion. Concentration profiles taken across the domain show a peak near the source depth $y = 0.005$ that decays downstream, with concentration increasing in time, and profiles along the channel show the highest values at $y = 0$ decaying with depth, consistent with the source position. Increasing the Peclet number from $Pe_x = 120$ to 650 and 1200 produces a more elongated patch with higher downstream concentration, owing to the growing dominance of advection over diffusion. These 2D results corroborate the 1D findings and confirm that the ADI scheme delivers stable, physically faithful solutions at low computational cost.

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5. CONCLUSION

A forward transport model for conservative pollutants in rivers under unsteady flow has been formulated, derived, and validated. The ADE was obtained from conservation of mass; a closed-form 1D analytical solution was constructed by successive transformations and Laplace transforms; an FTCSCS numerical scheme was developed with an explicit von Neumann stability condition; and an unconditionally stable, second-order 2D ADI scheme was derived with full x- and y-sweep formulations solved by the Thomas algorithm. The analytical and numerical 1D solutions agree closely, with RMSE falling from 0.028 at $Pe \ll 1$ to 0.004 at $Pe \gg 1$. Simulations show concentration highest near the source and decaying downstream, growing in time, and becoming increasingly advection-skewed as Pe rises; the four temporal coefficient regimes were quantified, with concentration lowest when both coefficients increase and highest when dispersion increases in a decelerating flow. By representing temporal variation in the flow parameters, the model is more faithful to natural rivers than constant- or spatially-varying-coefficient formulations, and it provides the accurate, efficient concentration fields on which inverse source-identification methods depend. Future extensions include reactive (non-conservative) pollutants through a reaction term, transient-storage (dead-zone) effects, and curvilinear domains for meandering rivers.

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